

# ON BURENKOV'S EXTENSION OPERATOR PRESERVING SOBOLEV-MORREY SPACES ON LIPSCHITZ DOMAINS

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**ABSTRACT.** We prove that Burenkov's Extension Operator preserves Sobolev spaces built on general Morrey spaces, including classical Morrey spaces. The analysis concerns bounded and unbounded open sets with Lipschitz boundaries in the  $n$ -dimensional Euclidean space.

## 1. INTRODUCTION

The extension problem is a classical problem in the theory of function spaces with important applications in many fields of mathematical analysis, in particular harmonic analysis and the theory of partial differential equations. Broadly speaking, the problem consists in extending to the whole of  $\mathbb{R}^n$  the elements of a space of functions defined on a given subset of  $\mathbb{R}^n$ , with preservation of certain differentiability and summability properties. The analysis of such problem goes back to the works of Whitney [9, 10] and Hestenes [5] who considered spaces of continuously differentiable functions. In the case of Sobolev spaces  $W^{l,p}(\Omega)$ , with  $l \in \mathbb{N}$  and  $p \in [1, \infty]$ , defined on open sets  $\Omega$  in  $\mathbb{R}^n$  with minimal boundary regularity, i.e.  $\Omega$  in the Lipschitz class  $C^{0,1}$ , the problem received important contributions by Calderon [4], Stein [7, 8] and Burenkov [1, 2]. They constructed three different linear bounded extension operators from  $W^{l,p}(\Omega)$  to  $W^{l,p}(\mathbb{R}^n)$ . Compared with the classical extension operator by Hestenes [5], the main striking feature of Calderon's, Stein's and Burenkov's operators consists in the fact that  $\Omega$  is not required to be of class  $C^l$  with  $l > 1$ . For a discussion concerning the differences between those operators, as well as for historical remarks and other references, we refer to Burenkov [2, 3], and to the earlier Stein's book [8]. For the convenience of the reader, we briefly mention here the main properties of such operators. Calderon's Extension Operator is based on an integral representation formula involving singular integral operators, hence it does not allow to deal with the cases  $p = 1, \infty$ . Stein's Extension Operator concerns all exponents  $p \in [1, \infty]$  and is universal in the sense that the same operator can be used for all orders of smoothness

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$l$ . Burenkov's Extension Operator is not universal but allows dealing with all exponents  $p \in [1, \infty]$ , and also with anisotropic Sobolev spaces. Moreover, Burenkov's operator provides functions which are  $C^\infty$  outside  $\Omega$  and the order of growth of the derivatives of such functions, when approaching the boundary, is the best possible in some sense. We also mention that Burenkov's operator allows to deal with open sets of class  $C^{0,\gamma}$  with  $\gamma < 1$ , in which case the target space is not  $W^{l,p}(\mathbb{R}^n)$  but  $W^{\gamma l,p}(\mathbb{R}^n)$ .

We denote Burenkov's Extension Operator by  $T$  and we refer to formula (4) for its definition in the case of an elementary Lipschitz domain  $\Omega$  given by the subgraph of a Lipschitz function, and to formula (25) for the case of general bounded or unbounded Lipschitz domains. For simplicity, we do not emphasize the dependence of  $T$  on  $l$  in the notation, but it is always understood. Burenkov's Extension Operator is also described in great detail in Burenkov's book [3, Chap. 6], to which we shall refer in this paper for any result required in our proofs.

We note that the operator  $T$  in (4) is defined by means of a sequence of mollifiers with variable steps and has a local nature in the sense that the values of the extended function  $Tf$  around a point in  $\mathbb{R}^n \setminus \Omega$  depend only on the values of  $f$  localized around certain 'reflected' points inside  $\Omega$ .

The main aim of the present paper is to exploit the local nature of Burenkov's Extension Operator in order to prove that such operator preserves also Sobolev-Morrey spaces.

Given  $p \in [1, \infty[$ , a function  $\phi : ]0, \infty[ \rightarrow ]0, \infty[$  and  $\delta \in ]0, \infty[$ , for all  $f \in L_{loc}^p(\Omega)$ , we set

$$\|f\|_{M_p^{\phi,\delta}(\Omega)} := \sup_{x \in \Omega} \sup_{0 < r < \delta} \left( \frac{1}{\phi(r)} \int_{B_r(x) \cap \Omega} |f|^p dy \right)^{\frac{1}{p}}.$$

We also write  $M_p^\phi(\Omega)$  to denote  $M_p^{\phi,\infty}(\Omega)$ . The Morrey space  $M_p^{\phi,\delta}(\Omega)$  is the space of functions  $f \in L_{loc}^p(\Omega)$  such that  $\|f\|_{M_p^{\phi,\delta}(\Omega)} < \infty$ . If  $\phi(r) = r^\lambda$  for all  $r \in ]0, \infty[$ , we obtain the classical Morrey spaces  $M_p^{\lambda,\delta}(\Omega)$  (which are known to be of interest only in the case  $\lambda \in [0, n]$ ).

Given  $l \in \mathbb{N}$ ,  $p \in [1, \infty[$  and  $\delta \in [0, \infty[$ , the main result of the paper is the following estimate

$$(1) \quad \|D^\alpha T f\|_{M_p^{\phi,\delta}(\mathbb{R}^n)} \leq c \sum_{0 \leq |\beta| \leq |\alpha|} \|D^\beta f\|_{M_p^{\phi,\delta}(\Omega)},$$

for all  $f \in W^{l,p}(\Omega)$  and  $|\alpha| \leq l$ , where  $c > 0$  is independent of  $f$ . See Theorem 3.1. Moreover, we also prove that if  $\Omega$  is a bounded or an elementary unbounded domain, then  $c$  can be chosen to be independent of  $\delta$ , hence in these cases estimate (1) holds also if  $\delta = \infty$ . See Corollary 2.1 for the case of elementary unbounded domains.

In particular, if  $f \in W^{l,p}(\Omega)$  is such that  $D^\alpha f \in M_p^{\phi,\delta}(\Omega)$  for all  $|\alpha| \leq l$  then  $D^\alpha T f \in M_p^{\phi,\delta}(\Omega)$  for all  $|\alpha| \leq l$ .

This paper is organized as follows. In section 2, we consider the case of elementary Lipschitz domains defined by the subgraphs of Lipschitz continuous functions. Section 3 is devoted to the case of general Lipschitz open sets.

For another contribution in this field of investigation, we refer to Khidr and Yeihia [6] which obtain results radically different from ours.

## 2. BURENKOV'S EXTENSION OPERATOR ON ELEMENTARY LIPSCHITZ DOMAINS

In this paper the elements of  $\mathbb{R}^n$  are denoted by  $x = (\bar{x}, x_n)$  with  $\bar{x} \in \mathbb{R}^{n-1}$  and  $x_n \in \mathbb{R}$ .

If  $\Omega$  is an open subset of  $\mathbb{R}^n$ , we denote by  $W^{l,p}(\Omega)$  the Sobolev space of function  $f \in L^p(\Omega)$  with weak derivatives  $D^\alpha f \in L^p(\Omega)$  for all  $|\alpha| \leq l$ , endowed with the norm  $\|f\|_{W^{l,p}(\Omega)} = \sum_{0 \leq |\alpha| \leq l} \|D^\alpha f\|_{L^p(\Omega)}$ .

**2.1. The case of unbounded Lipschitz subgraphs.** In this subsection we consider elementary Lipschitz domains  $\Omega$  in  $\mathbb{R}^n$  of the form

$$(2) \quad \Omega = \{x = (\bar{x}, x_n) \in \mathbb{R}^n : x_n < \varphi(\bar{x}), \bar{x} \in \mathbb{R}^{n-1}\},$$

where  $\varphi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  is a Lipschitz continuous function, that is there exists a positive constant  $M$  such that

$$(3) \quad |\varphi(\bar{x}) - \varphi(\bar{y})| \leq M|\bar{x} - \bar{y}|, \quad \forall \bar{x}, \bar{y} \in \mathbb{R}^{n-1}.$$

The best constant  $M$  in inequality (3) is the Lipschitz constant of  $\varphi$  and is denoted by  $\text{Lip}\varphi$ .

Let  $G = \mathbb{R}^n \setminus \bar{\Omega}$ . For every  $k \in \mathbb{Z}$ , we set

$$G_k = \{x \in G : 2^{-k-1} < \rho_n(x) \leq 2^{-k}\}$$

where  $\rho_n(x) = x_n - \varphi(\bar{x})$  is the signed distance from  $x \in \mathbb{R}^n$  to  $\partial G$  in the  $x_n$  direction. Clearly,  $\rho_n(x) \geq 0$  for all  $x \in G$ .

In the sequel, we need the following Partition of Unity's Lemma from [3, Lemma 18]. Here  $\mathbb{N}_0$  denotes the set of natural numbers including zero.

**Lemma 2.1.** *There exists a sequence of nonnegative functions  $\psi_k$  belonging to  $C^\infty(\mathbb{R}^n)$ , for all  $k \in \mathbb{Z}$ , satisfying the following conditions:*

- (i)  $\sum_{k=-\infty}^{\infty} \psi_k = \begin{cases} 1, & \text{if } x \in G, \\ 0, & \text{if } x \notin G; \end{cases}$
- (ii)  $G = \bigcup_{k=-\infty}^{\infty} \text{supp}\psi_k$  and the covering  $\{\text{supp}\psi_k\}_{k \in \mathbb{Z}}$  has multiplicity equal to 2;
- (iii)  $G_k \subset \text{supp}\psi_k \subset G_{k-1} \cup G_k \cup G_{k+1}$ , for all  $k \in \mathbb{Z}$ ;
- (iv)  $|D^\alpha \psi_k(x)| \leq c(\alpha)2^{k|\alpha|}$ , for all  $x \in \mathbb{R}^n$ ,  $k \in \mathbb{Z}$ ,  $\alpha \in \mathbb{N}_0^n$ .

We are now ready to recall the definition of Burenkov's Extension Operator for an elementary Lipschitz domain  $\Omega$  as in (2).

Let  $l \in \mathbb{N}$  and  $1 \leq p \leq \infty$ . For every  $f \in W^{l,p}(\Omega)$ , we set

$$(4) \quad (Tf)(x) = \begin{cases} f(x), & \text{if } x \in \Omega, \\ \sum_{k=-\infty}^{\infty} \psi_k(x) f_k(x), & \text{if } x \in G, \end{cases}$$

where

$$\begin{aligned} f_k(x) &= \int_{\mathbb{R}^n} f(\bar{x} - 2^{-k}\bar{z}, x_n - A2^{-k}z_n) \omega(z) dz = \\ &= A^{-1}2^{kn} \int_{\mathbb{R}^n} \omega(2^k(\bar{x} - \bar{y}), A^{-1}2^k(x_n - y_n)) f(y) dy, \end{aligned}$$

$A = 16(M+1)$ , and  $\omega \in C_c^\infty(\mathbb{R}^n)$  is a kernel mollification satisfying

$$\text{supp } \omega \subset \left\{ x \in \overline{B(0,1)} : x_n \geq \frac{1}{2} \right\},$$

$$\int_{B(0,1)} \omega(z) dz = 1 \text{ and } \int_{B(0,1)} \omega(z) z^\alpha dx = 0, \alpha \in \mathbb{N}_0^n, 0 < |\alpha| \leq l.$$

The operator  $T$  is a linear continuous operator from  $W^{l,p}(\Omega)$  to  $W^{l,p}(\mathbb{R}^n)$ , see [3, p. 286].

Following [3], for every  $k \in \mathbb{Z}$  we set

$$\tilde{G}_k = G_{k-1} \cup G_k \cup G_{k+1} = \{x \in G : 2^{-k-2} < \rho_n(x) \leq 2^{-k+1}\}.$$

We prove now the following

**Lemma 2.2.** *Let  $B_r$  be a ball in  $\mathbb{R}^n$  of radius  $r$  such that  $B_r \cap G \neq \emptyset$ . Let  $h \in \mathbb{Z}$  be the minimum integer such that  $B_r \cap G_h \neq \emptyset$ . Let  $k \in \mathbb{Z}$  be such that  $k \geq h+3$  and  $B_r \cap \tilde{G}_k \neq \emptyset$ . Then*

$$|2^{-(h+3)} - 2^{-k}| \leq r(M+1),$$

where  $M$  is in (3).

*Proof.* Since  $B_r \cap G_h, B_r \cap \tilde{G}_k \neq \emptyset$  and  $k \geq h+3$ , it follows that  $\{x \in B_r : \rho_n(x) = 2^{-h-2}\}, \{x \in B_r : \rho_n(x) = 2^{-k+1}\} \neq \emptyset$ . Hence, by taking  $y, w \in B_r$  with  $y_n - \varphi(\bar{y}) = 2^{-h-2}$  and  $w_n - \varphi(\bar{w}) = 2^{-k+1}$  we have

$$\begin{aligned} |2^{-(h+3)} - 2^{-k}| &= \frac{1}{2} |2^{-h-2} - 2^{-k+1}| = \frac{1}{2} |y_n - \varphi(\bar{y}) - w_n + \varphi(\bar{w})| \\ &\leq \frac{1}{2} (|y_n - w_n| + M|\bar{y} - \bar{w}|) \leq r(M+1). \end{aligned}$$

□

Then we need the following lemma. Here and in the sequel by  $B_r(x)$  we denote a ball with center  $x$  and radius  $r$ . Moreover,  $M$  is the constant in (3).

**Lemma 2.3.** *Let  $B_r$  and  $h \in \mathbb{Z}$  be as in Lemma 2.2, and  $E > 0$ . Then there exists a positive constant  $S$  depending only on  $M, E$  such that for every  $\eta \in \mathbb{R}^n$ , with  $|\eta| < E$ , there exists a ball  $B_{Sr}(x_\eta)$ , such that*

$$(5) \quad \bigcup_{k=h+3}^{\infty} (B_r \cap \tilde{G}_k - 2^{-k}\eta) \subset B_{Sr}(x_\eta).$$

Moreover, there exist  $K \in \mathbb{N}$  depending only on  $n, M, E$ , and  $K$  balls  $B_r(x_\eta^{(i)})$ ,  $i = 1, \dots, K$ , such that

$$(6) \quad \bigcup_{k=h+3}^{\infty} (B_r \cap \tilde{G}_k - 2^{-k}\eta) \subset \bigcup_{i=1}^K B_r(x_\eta^{(i)}).$$

Finally,  $x_\eta$  and  $x_\eta^{(i)}$  can be chosen to depend with continuity on  $\eta$  for all  $i = 1, \dots, K$ .

*Proof.* We suppose directly that  $B_r \cap \tilde{G}_{h+3} \neq \emptyset$ , otherwise the unions in the left hand-sides of (5) and (6) are empty, and the statement is trivial. Let  $k \geq h+3$  be such that  $B_r \cap \tilde{G}_k \neq \emptyset$ . Let  $a \in B_r \cap \tilde{G}_{h+3}$  and  $b \in B_r \cap \tilde{G}_k$ . Then by Lemma 2.2, for all  $\eta \in \mathbb{R}^n$ , with  $|\eta| < E$ , we have

$$\begin{aligned} |b - 2^{-k}\eta - (a - 2^{-(h+3)}\eta)| &\leq |b - a| + |2^{-k} - 2^{-(h+3)}||\eta| \\ &\leq [2 + (M+1)E]r. \end{aligned}$$

Then, choosing  $S = 2[2 + (M+1)E]$ , the ball  $B_{Sr}(x_\eta)$ , with radius  $Sr$  and center  $x_\eta = a - 2^{-(h+3)}\eta$ , satisfies inclusion (5).

Finally, it is obvious that each ball  $B_{Sr}(x_\eta)$  can be covered by a finite number of balls of radius  $r$  as in (6). Moreover, the fact that the centers  $x_\eta^{(i)}$  can be chosen to depend with continuity on  $\eta$  can be deduced by the continuous dependence on  $\eta$  of  $x_\eta = a - 2^{-(h+3)}\eta$  via a simple but lengthy argument which is not worth including here.  $\square$

As in [3, Chap. 6], for every  $k \in \mathbb{Z}$  we set

$$\tilde{\Omega}_k = \{x \in \Omega : 2^{-k-2} < |\rho_n(x)| \leq b2^{-k+1}\},$$

where  $b = 10A$ .

We can now prove the following lemma.

In the proof of the following lemma and of the other statements in the sequel, the value of the constant  $c$  may change from line to line but is always independent of the function  $f$  and of the radius  $r$ .

**Lemma 2.4.** *Let  $f \in W^{l,p}(\Omega)$  and  $B_r$  a ball in  $\mathbb{R}^n$  of radius  $r$  such that  $B_r \cap G \neq \emptyset$ . The following statements hold:*

- (i) *There exists  $c > 0$  depending only on  $n, l, p, M, \omega$  and there exists  $H \in \mathbb{N}$ , depending only on  $n$  and  $M$  such that for every  $z \in B_1(0)$ , with  $z_n > 1/2$ , there exist  $H$  balls  $B_r(x_z^{(i)})$ ,  $i = 1, \dots, H$ , such that*

$$(7) \quad \|D^\alpha f_k\|_{L^p(B_r \cap \tilde{G}_k)} \leq c \int_{\{z \in B_1(0): z_n > 1/2\}} \|D^\alpha f\|_{L^p(\cup_{i=1}^H B_r(x_z^{(i)}) \cap \tilde{\Omega}_k)} dz,$$

*for all  $k \in \mathbb{Z}$ , and  $\alpha \in \mathbb{N}_0^n$ , with  $|\alpha| \leq l$ .*

- (ii) *Let  $\mathcal{U} \subset \mathbb{R}^n$  be a fixed measurable set and let  $d = \sup\{\rho_n(x) : x \in B_r \cap \mathcal{U}\}$ . Assume that  $d < \infty$ . There exists  $c > 0$  depending only on  $n, l, p, M, \omega$ , there exists  $H_{\mathcal{U}} \in \mathbb{N}$  depending only on  $n, M, d$ , and for every  $\alpha \in \mathbb{N}_0^n$  with  $|\alpha| \leq l$  there exists a function  $g_\alpha$  independent of  $r, \mathcal{U}$ , such that for every  $z \in B_{1+cd}(0)$  there exist  $H_{\mathcal{U}}$  balls  $B_r(x_z^{(i)})$ ,  $i = 1, \dots, H_{\mathcal{U}}$ , such that*

$$\begin{aligned} \|D^\alpha f_k - g_\alpha\|_{L^p(B_r \cap \mathcal{U} \cap \tilde{G}_k)}^p &\leq c 2^{pk(|\alpha|-l)} \int_{B_{1+cd}(0)} \sum_{|\beta|=l} \|D^\beta f\|_{L^p(\cup_{i=1}^{H_{\mathcal{U}}} B_r(x_z^{(i)}) \cap \tilde{\Omega}_k)}^p dz, \end{aligned}$$

*for all  $k \in \mathbb{Z}$ .*

Moreover, in both statements points  $x_z^{(i)}$  can be chosen to depend with continuity on  $z$ .

*Proof.* We begin with proving statement (i). By differentiating under integral sign and using Minkowskii inequality we get

$$\begin{aligned} (8) \quad \|D^\alpha f_k\|_{L^p(B_r \cap \tilde{G}_k)} &\leq c \int_{\{z \in B_1(0): z_n > 1/2\}} \|D^\alpha f(\bar{x} - 2^{-k}\bar{z}, x_n - A2^{-k}z_n)\|_{L^p(B_r \cap \tilde{G}_k)} dz \\ &= c \int_{\{z \in B_1(0): z_n > 1/2\}} \|D^\alpha f\|_{L^p(B_r \cap \tilde{G}_k - 2^{-k}(\bar{z}, Az_n))} dz. \end{aligned}$$

Let  $h \in \mathbb{Z}$  be the minimum integer such that  $B_r \cap G_h \neq \emptyset$ . By Lemma 2.3, there exists  $K \in \mathbb{N}$  depending only on  $n$  and  $M$  such that for every  $z \in \mathbb{R}^n$ ,

$|z| < 1$  there exist  $K$  balls  $B_r(x_z^{(i)})$  such that

$$\bigcup_{k \in \mathbb{Z}} (B_r \cap \tilde{G}_k - 2^{-k}(\bar{z}, Az_n)) \subset \bigcup_{k=h-1}^{h+2} (B_r \cap \tilde{G}_k - 2^{-k}(\bar{z}, Az_n)) \bigcup_{i=1}^K B_r(x_z^{(i)}),$$

hence

$$(9) \quad \bigcup_{k \in \mathbb{Z}} (B_r \cap \tilde{G}_k - 2^{-k}(\bar{z}, Az_n)) \subset \bigcup_{i=1}^H B_r(x_z^{(i)}),$$

where we have set  $H = K + 4$  and  $B_r(x_z^{(K+j)}) = B_r - 2^{-(h+j)}(\bar{z}, Az_n)$  for  $j = -1, 0, 1, 2$ .

Now we observe that for all  $k \in \mathbb{Z}$  and  $z \in B_1(0)$  with  $z_n > 1/2$  we have

$$(10) \quad \tilde{G}_k - 2^{-k}(\bar{z}, Az_n) \subseteq \tilde{\Omega}_k.$$

Indeed, if  $x \in \tilde{G}_k$ , that is  $2^{-k-2} < x_n - \varphi(\bar{x}) \leq 2^{-k+1}$ , we have

$$\begin{aligned} \varphi(\bar{x} - 2^{-k}\bar{z}) - x_n + 2^{-k}Az_n &= \varphi(\bar{x} - 2^{-k}\bar{z}) - \varphi(\bar{x}) + \varphi(\bar{x}) - x_n + 2^{-k}Az_n \leq \\ &\leq M2^{-k}|\bar{z}| - 2^{-k-2} + 2^{-k}Az_n \leq M2^{-k} - 2^{-k-2} + 2^{-k}A < b2^{-k+1}, \end{aligned}$$

and

$$\begin{aligned} \varphi(\bar{x} - 2^{-k}\bar{z}) - x_n + 2^{-k}Az_n &= \varphi(\bar{x} - 2^{-k}\bar{z}) - \varphi(\bar{x}) + \varphi(\bar{x}) - x_n + 2^{-k}Az_n > \\ &> -M2^{-k}|\bar{z}| - 2^{-k+1} + 2^{-k}Az_n \geq -M2^{-k} - 2^{-k+1} + 2^{-k-1}A > 2^{-k-2}. \end{aligned}$$

By (9) and (10) we deduce that

$$B_r \cap \tilde{G}_k - 2^{-k}(\bar{z}, Az_n) \subset \bigcup_{i=1}^H B_r(x_z^{(i)}) \cap \tilde{\Omega}_k$$

which, combined with (8), proves the validity of (7).

We now prove statement (ii). By differentiating under integral sign, changing variables and integrating by parts, we get

$$(11) \quad D^\alpha f_k(x) = A^{-\alpha_n} 2^{k|\alpha|} \int_{B_1(0)} f(\bar{x} - 2^{-k}\bar{z}, x_n - A2^{-k}z_n) D^\alpha w(z) dz.$$

We set  $x^* = (\bar{x}, x_n - \frac{9}{4}A\rho_n(x))$ ,  $\tilde{x} = (\bar{x} - 2^{-k}\bar{z}, x_n - A2^{-k}z_n)$  and we denote by  $V_{\tilde{x}}$  the conic body with vertex in  $\tilde{x}$  constructed on the ball  $B_{4\rho_n(x)}(x^*)$ , i.e.,  $V_{\tilde{x}} = \cup_{y \in B_{4\rho_n(x)}(x^*)} (x^*, y)$  (where  $(x^*, y)$  is the 'open' segment joining  $x^*$  and  $y$ ). Let  $\mu \in C_c^\infty(B_1(0))$  be such that  $\int_{B_1(0)} \mu dx = 1$ ,

and  $w_x(y) = (4\rho_n(x))^{-n} \mu\left(\frac{x^*-y}{4\rho_n(x)}\right)$ . By the Sobolev Integral Representation Formula (cf. [3, Theorems 4, 5, Chap. 3]), we get

$$(12) \quad f(\tilde{x}) = P(\tilde{x}, x) + \sum_{|\gamma|=l} r_\gamma(\tilde{x}, x),$$

where

$$P(\tilde{x}, x) = \int_{B_{4\rho_n(x)}(x^*)} \sum_{|\gamma| \leq l} \frac{1}{\gamma!} D^\gamma f(y) (\tilde{x} - y)^\gamma w_x(y) dy$$

and

$$r_\gamma(\tilde{x}, x) = \int_{V_{\tilde{x}}} \frac{D^\gamma f(y)}{|\tilde{x} - y|^{n-l}} w_{\gamma,x}(y) dy,$$

where  $w_{\gamma,x}$  is the appropriate kernel associated with  $\omega$  appearing in the formula as in [3, (3.38)]. By (11) and (12) we deduce that

$$\begin{aligned} D^\alpha f_k(x) &= A^{-\alpha_n} 2^{k|\alpha|} \int_{B_1(0)} P(\tilde{x}, x) D^\alpha \omega(z) dz \\ &\quad + A^{-\alpha_n} 2^{k|\alpha|} \int_{B_1(0)} \sum_{|\gamma|=l} r_\gamma(\tilde{x}, x) D^\alpha \omega(z) dz. \end{aligned}$$

We set

$$g_\alpha(x) = A^{-\alpha_n} 2^{k|\alpha|} \int_{B_1(0)} P(\tilde{x}, x) D^\alpha \omega(z) dz.$$

We note that function  $g_\alpha$  does not depend on  $k$  (see [3, p. 280]).

We now estimate

$$\begin{aligned} (13) \quad \|D^\alpha f_k - g_\alpha\|_{L^p(B_r \cap \mathcal{U} \cap \tilde{G}_k)} &= \left\| A^{-\alpha_n} 2^{k|\alpha|} \sum_{|\gamma|=l} \int_{B_1(0)} r_\gamma(\tilde{x}, x) D^\alpha \omega(z) dz \right\|_{L^p(B_r \cap \mathcal{U} \cap \tilde{G}_k)}. \end{aligned}$$

To do so, we proceed as follows

$$\begin{aligned} &\left\| \int_{B_1(0)} r_\gamma(\tilde{x}, x) D^\alpha \omega(z) dz \right\|_{L^p(B_r \cap \mathcal{U} \cap \tilde{G}_k)}^p \\ &\leq c \int_{B_r \cap \mathcal{U} \cap \tilde{G}_k} \left| \int_{B_1(0)} r_\gamma(\tilde{x}, x) dz \right|^p dx \\ &\leq c \int_{B_r \cap \mathcal{U} \cap \tilde{G}_k} \int_{B_1(0)} |r_\gamma(\tilde{x}, x)|^p dz dx. \end{aligned}$$



Using Minkowskii inequality and the fact that  $V_x \subset \tilde{\Omega}_k$  and that  $\text{diam} V_x \leq 20A2^{-k}$  (cf. [3, (6.83) and pp. 278-279]), we get

$$\begin{aligned}
& \left( \int_{B_1(0)} |r_\gamma(\tilde{x}, x)|^p dz \right)^{1/p} \\
&= \left( \int_{B_1(0)} \left| \int_{V_{\tilde{x}}} \frac{D^\gamma f(y) \chi_{\tilde{\Omega}_k}(y)}{|\tilde{x} - y|^{n-l}} w_{\gamma, x}(y) dy \right|^p dz \right)^{1/p} \\
&\leq c \left( \int_{B_1(0)} \left| \int_{B_{20A2^{-k}}(0)} \frac{D^\gamma f(\tilde{x} - \eta) \chi_{\tilde{\Omega}_k}(\tilde{x} - \eta)}{|\eta|^{n-l}} d\eta \right|^p dz \right)^{1/p} \\
&\leq c \int_{B_{20A2^{-k}}(0)} \left( \int_{B_1(0)} \frac{|D^\gamma f(\tilde{x} - \eta) \chi_{\tilde{\Omega}_k}(\tilde{x} - \eta)|^p}{|\eta|^{p(n-l)}} dz \right)^{1/p} d\eta \\
&= c \int_{B_{20A2^{-k}}(0)} \left( \int_{B_1(0)} |D^\gamma f(\tilde{x} - \eta) \chi_{\tilde{\Omega}_k}(\tilde{x} - \eta)|^p dz \right)^{1/p} \frac{d\eta}{|\eta|^{n-l}} \\
&= c \int_{B_{20A2^{-k}}(0)} \left( \int_{B_1(0) - \eta} |D^\gamma f(\tilde{x}) \chi_{\tilde{\Omega}_k}(\tilde{x})|^p dz \right)^{1/p} \frac{d\eta}{|\eta|^{n-l}} \\
&\leq c \int_{B_{20A2^{-k}}(0)} \left( \int_{B_{1+20A2^{-k}}(0)} |D^\gamma f(\tilde{x}) \chi_{\tilde{\Omega}_k}(\tilde{x})|^p dz \right)^{1/p} \frac{d\eta}{|\eta|^{n-l}} \\
&\leq c \left( \int_{B_{1+20A2^{-k}}(0)} |D^\gamma f(\tilde{x}) \chi_{\tilde{\Omega}_k}(\tilde{x})|^p dz \right)^{1/p} \int_{B_{20A2^{-k}}(0)} \frac{d\eta}{|\eta|^{n-l}} \\
&\leq c 2^{-kl} \left( \int_{B_{1+20A2^{-k}}(0)} |D^\gamma f(\tilde{x}) \chi_{\tilde{\Omega}_k}(\tilde{x})|^p dz \right)^{1/p},
\end{aligned}$$

from which it follows that

$$\begin{aligned}
(14) \quad & \left\| \int_{B_1(0)} r_\gamma(\tilde{x}, x) D^\alpha \omega(z) dz \right\|_{L^p(B_r \cap \mathcal{U} \cap \tilde{G}_k)}^p \\
&\leq c 2^{-klp} \int_{B_r \cap \mathcal{U} \cap \tilde{G}_k} \int_{B_{1+20A2^{-k}}(0)} |D^\gamma f(\tilde{x}) \chi_{\tilde{\Omega}_k}(\tilde{x})|^p dz dx \\
&\leq c 2^{-klp} \int_{B_{1+20A2^{-k}}(0)} \int_{B_r \cap \mathcal{U} \cap \tilde{G}_k} |D^\gamma f(\tilde{x}) \chi_{\tilde{\Omega}_k}(\tilde{x})|^p dx dz.
\end{aligned}$$

By (14) and (13) we obtain

$$\begin{aligned} & \|D^\alpha f_k - g_\alpha\|_{L^p(B_r \cap \mathcal{U} \cap \tilde{G}_k)}^p \\ & \leq c 2^{pk(|\alpha|-l)} \sum_{|\gamma|=l} \int_{B_{1+20A2^{-k}}(0)} \int_{B_r \cap \mathcal{U} \cap \tilde{G}_k} |D^\gamma f(\tilde{x}) \chi_{\tilde{\Omega}_k}(\tilde{x})|^p dx dz. \end{aligned}$$

If we denote by  $\bar{k}$  the minimum  $k \in \mathbb{Z}$  such that  $B_r \cap \mathcal{U} \cap \tilde{G}_k \neq \emptyset$ , then we easily see that  $20A2^{-\bar{k}} \leq cd$  and

$$\begin{aligned} (15) \quad & \|D^\alpha f_k - g_\alpha\|_{L^p(B_r \cap \mathcal{U} \cap \tilde{G}_k)}^p \\ & \leq c 2^{pk(|\alpha|-l)} \sum_{|\gamma|=l} \int_{B_{1+20A2^{-\bar{k}}}(0)} \int_{B_r \cap \mathcal{U} \cap \tilde{G}_k} |D^\gamma f(\tilde{x}) \chi_{\tilde{\Omega}_k}(\tilde{x})|^p dx dz \\ & \leq c 2^{pk(|\alpha|-l)} \sum_{|\gamma|=l} \int_{B_{1+cd}(0)} \int_{B_r \cap \mathcal{U} \cap \tilde{G}_k} |D^\gamma f(\tilde{x}) \chi_{\tilde{\Omega}_k}(\tilde{x})|^p dx dz. \end{aligned}$$

Now by Lemma 2.3 and its proof we obtain that for all  $z \in B_{1+cd}(0)$

$$\begin{aligned} (16) \quad & \bigcup_{k \in \mathbb{Z}} (B_r \cap \tilde{G}_k - 2^{-k}(\bar{z}, Az_n)) = \\ & \bigcup_{k=h-1}^{h+2} (B_r \cap \tilde{G}_k - 2^{-k}(\bar{z}, Az_n)) \cup \bigcup_{k=h+3}^{\infty} (B_r \cap \tilde{G}_k - 2^{-k}(\bar{z}, Az_n)) \\ & \subset \bigcup_{k=h-1}^{h+2} (B_r \cap \tilde{G}_k - 2^{-k}(\bar{z}, Az_n)) \cup B_{S'r}(x_z), \end{aligned}$$

where  $S' = 2 + (M+1)(1+cd)$  and  $B_{S'r}(x_z)$  is the ball provided by Lemma 2.3.

Clearly, as in the proof of Lemma 2.3, one can easily deduce by (16) the existence of  $H_{\mathcal{U}} \in \mathbb{N}$  and of  $H_{\mathcal{U}}$  balls  $B_r(x_z^{(i)})$ ,  $i = 1, \dots, H_{\mathcal{U}}$ , defined for all  $z \in B_{1+cd}(0)$  as in the statement such that

$$(17) \quad \bigcup_{k \in \mathbb{Z}} (B_r \cap \tilde{G}_k - 2^{-k}(\bar{z}, Az_n)) = \bigcup_{i=1}^{H_{\mathcal{U}}} B_r(x_z^{(i)}).$$

By (15) and (17) we get

$$\begin{aligned}
& \|D^\alpha f_k - g_\alpha\|_{L^p(B_r \cap \mathcal{U} \cap \tilde{G}_k)}^p \\
& \leq c 2^{pk(|\alpha|-l)} \sum_{|\gamma|=l} \int_{B_{1+cd}(0)} \|D^\gamma f \chi_{\tilde{\Omega}_k}\|_{L^p(\cup_{i=1}^{H_{\mathcal{U}}} B_r(x_z^{(i)}) \cap \tilde{\Omega}_k)}^p dz \\
& \leq c 2^{pk(|\alpha|-l)} \int_{B_{1+cd}(0)} \sum_{|\beta|=l} \|D^\beta f\|_{L^p(\cup_{i=1}^{H_{\mathcal{U}}} B_r(x_z^{(i)}) \cap \tilde{\Omega}_k)}^p dz.
\end{aligned}$$

□

Then we have the following

**Theorem 2.1.** *Let  $B_r$  a ball in  $\mathbb{R}^n$  with radius  $r$  such that  $B_r \cap G \neq \emptyset$ . The following statements hold:*

- (i) *There exist  $c > 0$ ,  $H \in \mathbb{N}$  and  $H$  balls  $B_r(x_z^{(i)})$ ,  $i = 1, \dots, H$ , as in Lemma 2.4 such that*

$$\|Tf\|_{L^p(B_r \cap G)}^p \leq c \int_{\{z \in B_1(0) : z_n > 1/2\}} \|f\|_{L^p(\cup_{i=1}^H B_r(x_z^{(i)}) \cap \Omega)}^p dz,$$

*for all  $f \in W^{l,p}(\Omega)$ .*

- (ii) *Let  $\mathcal{U} \subset \mathbb{R}^n$  be a fixed measurable set and let  $d = \sup\{\rho_n(x) : x \in B_r \cap \mathcal{U}\}$ . Assume that  $d < \infty$ . There exist  $c > 0$ ,  $H_{\mathcal{U}} \in \mathbb{N}$  and  $H_{\mathcal{U}}$  balls  $B_r(x_z^{(i)})$ ,  $i = 1, \dots, H_{\mathcal{U}}$ , such that for all  $\alpha \in \mathbb{N}_0^n$  with  $|\alpha| = l$*

$$(18) \quad \|D^\alpha T f\|_{L^p(B_r \cap \mathcal{U} \cap G)}^p \leq c \int_{B_{1+cd}(0)} \sum_{|\beta|=l} \|D^\beta f\|_{L^p(\cup_{i=1}^{H_{\mathcal{U}}} B_r(x_z^{(i)}) \cap \Omega)}^p dz,$$

*for all  $f \in W^{l,p}(\Omega)$ .*

*Proof.* We begin with statement (i). Since the multiplicity of the covering  $\{\text{supp} \psi_k\}_{k \in \mathbb{Z}}$  is equal to 2, by (7) with  $\alpha = 0$  and Hölder inequality, we get

$$\begin{aligned}
\|Tf\|_{L^p(B_r \cap G)}^p &= \int_{B_r \cap G} \left| \sum_{k=-\infty}^{+\infty} \psi_k(x) f_k(x) \right|^p dx \leq \\
&\leq 2^{p-1} \int_{B_r \cap G} \sum_{k=-\infty}^{+\infty} |\psi_k(x) f_k(x)|^p dx = 2^{p-1} \sum_{k=-\infty}^{+\infty} \int_{B_r \cap G} |\psi_k(x) f_k(x)|^p dx \\
&\leq 2^{p-1} \sum_{k=-\infty}^{+\infty} \int_{B_r \cap \tilde{G}_k} |\psi_k(x) f_k(x)|^p dx \leq 2^{p-1} \sum_{k=-\infty}^{+\infty} \int_{B_r \cap \tilde{G}_k} |f_k(x)|^p dx \\
&\leq c \int_{\{z \in B_1(0) : z_n > 1/2\}} \sum_{k=-\infty}^{+\infty} \|f\|_{L^p(\cup_{i=1}^H B_r(x_z^{(i)}) \cap \tilde{\Omega}_k)}^p dz \\
&\leq c \kappa_\Omega \int_{\{z \in B_1(0) : z_n > 1/2\}} \|f\|_{L^p(\cup_{i=1}^H B_r(x_z^{(i)}) \cap \Omega)}^p dz,
\end{aligned}$$

for all  $f \in W^{l,p}(\Omega)$ , where  $\kappa_\Omega$  denotes the multiplicity of the covering  $\{\tilde{\Omega}_k\}_{k \in \mathbb{Z}}$  of  $\Omega$ . Thus, in order to conclude it suffices to observe that  $\kappa_\Omega$  depends only on  $M$  by [3, Remark 12, Chap. 6].

We now prove statement (ii). Using statements (i) and (ii) in Lemma 2.4, statement (iv) in Lemma 2.1, and the simple equality  $\sum_{k \in \mathbb{Z}} D^{\alpha-\beta} \psi_k D^\beta f_k = \sum_{k \in \mathbb{Z}} D^{\alpha-\beta} \psi_k (D^\beta f_k - g_\beta)$  for  $\beta < \alpha$  (recall that  $g_\beta$  does not depend on  $k$ ) we get

$$\begin{aligned}
\|D^\alpha T f\|_{L^p(B_r \cap \mathcal{U} \cap G)}^p &= \int_{B_r \cap \mathcal{U} \cap G} \left| \sum_{\beta \leq \alpha} \frac{\alpha!}{(\alpha-\beta)! \beta!} \sum_{k \in \mathbb{Z}} D^{\alpha-\beta} \psi_k D^\beta f_k \right|^p dx \\
&\leq c \int_{B_r \cap \mathcal{U} \cap G} \left| \sum_{k \in \mathbb{Z}} \psi_k D^\alpha f_k \right|^p dx \\
&\quad + c \int_{B_r \cap \mathcal{U} \cap G} \left| \sum_{\beta < \alpha} \frac{\alpha!}{(\alpha-\beta)! \beta!} \sum_{k \in \mathbb{Z}} D^{\alpha-\beta} \psi_k D^\beta f_k \right|^p dx \\
&\leq c \sum_{k \in \mathbb{Z}} \|D^\alpha f_k\|_{L^p(B_r \cap \mathcal{U} \cap \tilde{G}_k)}^p + c \sum_{\beta < \alpha, k \in \mathbb{Z}} \|D^{\alpha-\beta} \psi_k (D^\beta f_k - g_\beta)\|_{L^p(B_r \cap \mathcal{U} \cap \tilde{G}_k)}^p
\end{aligned}$$

$$\begin{aligned}
&\leq c \int_{B_1(0)} \sum_{k=-\infty}^{+\infty} \|D^\alpha f\|_{L^p(\cup_{i=1}^H B_r(x_z^{(i)}) \cap \tilde{\Omega}_k)}^p dz \\
&\quad + c \sum_{\beta < \alpha, k \in \mathbb{Z}} 2^{k|\alpha-\beta|p} \|D^\beta f_k - g_\beta\|_{L^p(B_r \cap \mathcal{U} \cap \tilde{G}_k)}^p \\
&\leq c \kappa_\Omega \int_{B_1(0)} \|D^\alpha f\|_{L^p(\cup_{i=1}^{H\mathcal{U}} B_r(x_z^{(i)}) \cap \tilde{\Omega})}^p dz \\
&\quad + c \sum_{\beta < \alpha, k \in \mathbb{Z}} 2^{pk|\alpha-\beta|} 2^{pk(|\beta|-l)} \int_{B_{1+cd}(0)} \sum_{|\beta|=l} \|D^\beta f\|_{L^p(\cup_{i=1}^{H\mathcal{U}} B_r(x_z^{(i)}) \cap \tilde{\Omega}_k)}^p dz \\
&\leq c \kappa_\Omega \int_{B_1(0)} \|D^\alpha f\|_{L^p(\cup_{i=1}^{H\mathcal{U}} B_r(x_z^{(i)}) \cap \Omega)}^p dz \\
&\quad + c \kappa_\Omega \int_{B_{1+cd}(0)} \sum_{|\beta|=l} \|D^\beta f\|_{L^p(\cup_{i=1}^{H\mathcal{U}} B_r(x_z^{(i)}) \cap \Omega)}^p dz,
\end{aligned}$$

which allows to conclude.  $\square$

**Corollary 2.1.** *Let  $l \in \mathbb{N}_0$ ,  $p \in [1, \infty[$ ,  $\phi : ]0, +\infty[ \rightarrow ]0, +\infty[$ . The following statements hold:*

(i) *There exists  $c > 0$  such that*

$$(19) \quad \|Tf\|_{M_p^{\phi, \delta}(\mathbb{R}^n)} \leq c \|f\|_{M_p^{\phi, \delta}(\Omega)},$$

*for all  $\delta \in ]0, \infty]$  and  $f \in W^{l,p}(\Omega)$ .*

(ii) *Let  $D > 0$ . There exists  $c > 0$  such that*

$$(20) \quad \|D^\alpha Tf\|_{M_p^{\phi, \delta}(\mathbb{R}^n)} \leq c \sum_{|\beta|=l} \|D^\beta f\|_{M_p^{\phi, \delta}(\Omega)},$$

*for all  $\delta \in ]0, \infty]$ ,  $f \in W^{l,p}(\Omega)$  with  $\text{supp } f \subset \{x \in \Omega : |\rho_n(x)| < D\}$ , and  $\alpha \in \mathbb{N}_0^n$  with  $|\alpha| = l$ .*

*Proof.* Let  $\delta \in ]0, \infty]$  be fixed and let  $B_r$  be a ball in  $\mathbb{R}^n$  with radius  $r < \delta$ .

We begin with proving statement (i). Then by Theorem 2.1 (i) we have

$$\begin{aligned}
\|Tf\|_{L^p(B_r)}^p &= \|f\|_{L^p(B_r \cap \Omega)}^p + \|Tf\|_{L^p(B_r \cap G)}^p \\
&\leq \phi(r) \|f\|_{M_p^{\phi, \delta}(\Omega)}^p + c \int_{\{z \in B_1(0) : z_n > 1/2\}} \|f\|_{L^p(\cup_{i=1}^H B_r(x_z^{(i)}) \cap \Omega)}^p dz \\
&\leq \phi(r) \|f\|_{M_p^{\phi, \delta}(\Omega)}^p + c \phi(r) \int_{\{z \in B_1(0) : z_n > 1/2\}} \|f\|_{M_p^{\phi, \delta}(\Omega)}^p dz,
\end{aligned}$$

for all  $f \in W^{l,p}(\Omega)$ , which provides the validity of (19).

We now prove statement (ii). Let  $f$  be as in statement (ii). We claim that  $Tf(x) = 0$  for all  $x \in G$  with  $\rho_n(x) > 8D$ . Indeed, assume that  $x \in G$

with  $\rho_n(x) > 8D$  and let  $k \in \mathbb{Z}$  be such that  $x \in \tilde{G}_k$ . By [3, Remark 11 and (6.83)], it follows that the value of  $f_k(x)$  depends only on the values of  $f|_{\tilde{\Omega}_k}$  and in particular, if  $f|_{\tilde{\Omega}_k} = 0$  then  $f_k(x) = 0$ . Since  $\rho_n(x) > 8D$  and  $x \in \tilde{G}_k$ , we have that  $2^{-k+1} > 8D$ , hence  $2^{-k-2} > D$  which implies that  $\tilde{\Omega}_k \subset \{y \in \Omega : |\rho_n(y)| > D\} \subset (\text{supp } f)^c$ . Thus,  $f_k(x) = 0$  as we have claimed. We set

$$G_D = \{x \in G : \rho_n(x) \leq 8D\}.$$

Since  $\text{supp } Tf \subset G_D$ , we have

$$\begin{aligned} (21) \quad \|D^\alpha Tf\|_{L^p(B_r)}^p &= \|D^\alpha f\|_{L^p(B_r \cap \Omega)}^p + \|D^\alpha Tf\|_{L^p(B_r \cap G)}^p \\ &= \|D^\alpha f\|_{L^p(B_r \cap \Omega)}^p + \|D^\alpha Tf\|_{L^p(B_r \cap G_D)}^p. \end{aligned}$$

By applying (18) with  $\mathcal{U} = G_D$  and observing that  $d = \sup\{\rho_n(x) : x \in B_r \cap \mathcal{U}\} \leq 8D$ , by (21) we get

$$\begin{aligned} &\|D^\alpha Tf\|_{L^p(B_r)}^p \\ &\leq c\phi(r) \|D^\alpha f\|_{M_p^{\phi,\delta}(\Omega)}^p + c \int_{B_{1+cd}(0)} \sum_{|\beta|=l} \|D^\beta f\|_{L^p(\cup_{i=1}^H B_r(x_z^{(i)}) \cap \Omega)}^p dz \\ &\leq c\phi(r) \|D^\alpha f\|_{M_p^{\phi,\delta}(\Omega)}^p + c\phi(r) \int_{B_{1+8cD}(0)} \sum_{|\beta|=l} \|D^\beta f\|_{M_p^{\phi,\delta}(\Omega)}^p dz \end{aligned}$$

which implies (20). □

**2.2. The case of bounded Lipschitz subgraphs.** In this subsection, we consider bounded elementary domains with Lipschitz boundaries. Namely, these domains are bounded Lipschitz subgraphs. However, in order to treat the case of general Lipschitz domains in the next section, we need to take into account a number of parameters describing the size of such subgraphs. For this reason, following [3], it is convenient to give the following definition.

**Definition 2.1.** *Let  $d, D > 0$ ,  $M \geq 0$ . We say an open set  $\mathcal{H}$  in  $\mathbb{R}^n$  is a bounded elementary domain with Lipschitz boundary and parameters  $d, D$ ,*

$M$  if  $\mathcal{H}$  can be represented as

$$(22) \quad \mathcal{H} = \{x \in \mathbb{R}^n : \bar{x} \in W, a_n < x_n < \varphi(\bar{x})\}$$

where  $W = \prod_{i=1}^{n-1} ]a_i, b_i[$ ,  $-\infty < a_i < b_i < \infty$  for all  $i = 1, \dots, n$ ,  $\text{diam} \mathcal{H} < D$  and  $\varphi : W \rightarrow \mathbb{R}$  is a Lipschitz function such that

$$a_n + d < \varphi \text{ and } \text{Lip} \varphi \leq M.$$

We note that if  $\varphi$  is a Lipschitz function as in Definition 2.1, then  $\varphi$  can be extended to the whole of  $\mathbb{R}^{n-1}$  by means of a Lipschitz function  $F_\varphi$  such that  $\text{Lip} F_\varphi = \text{Lip} \varphi$ . In particular, given an elementary domain  $\mathcal{H}$  represented as in (22), we can define the following open set

$$\Omega_{\mathcal{H}} = \{x \in \mathbb{R}^n : x_n < F_\varphi(\bar{x})\}.$$

We find it convenient to set

$$\widetilde{W}^{l,p}(\mathcal{H}) = \{f \in W^{l,p}(\mathcal{H}) : \text{supp} f \subset \prod_{i=1}^n ]a_i, b_i[ \}.$$

Given a function  $f \in \widetilde{W}^{l,p}(\mathcal{H})$  then the extension-by-zero  $f_0$  of  $f$  (defined by  $f_0(x) = f(x)$  for all  $x \in \mathcal{H}$  and  $f_0(x) = 0$  for all  $x \in \Omega_{\mathcal{H}}$ ) belongs to  $W^{l,p}(\Omega_{\mathcal{H}})$ , because the distance of  $\text{supp} f$  from the boundary of  $\prod_{i=1}^n ]a_i, b_i[$  is positive, hence a standard truncation argument is applicable. Consider now the extension operator  $T$  defined by (4) for the open set  $\Omega = \Omega_{\mathcal{H}}$ . For all functions  $f \in \widetilde{W}^{l,p}(\mathcal{H})$  we set

$$(23) \quad T_{\mathcal{H}} f = T f_0.$$

It is clear that  $T_{\mathcal{H}} f \in W^{l,p}(\mathbb{R}^n)$  for all  $f \in \widetilde{W}^{l,p}(\mathcal{H})$ . The following theorem is an easy consequence of Corollary 2.1.

**Theorem 2.2.** *Let  $l \in \mathbb{N}_0$ ,  $p \in [1, \infty[$  and  $\phi : ]0, +\infty[ \rightarrow ]0, +\infty[$ . Let  $\mathcal{H}$  be a bounded elementary domain with Lipschitz boundary and parameters  $d, D, M$ . Then there exists  $c > 0$  depending only on  $n, l, p, D, M$  such that*

$$\|T_{\mathcal{H}} f\|_{M_p^{\phi,\delta}(\mathbb{R}^n)} \leq c \|f\|_{M_p^{\phi,\delta}(\mathcal{H})},$$

and

$$(24) \quad \|D^\alpha T_{\mathcal{H}} f\|_{M_p^{\phi,\delta}(\mathbb{R}^n)} \leq c \sum_{|\beta|=l} \|D^\beta f\|_{M_p^{\phi,\delta}(\mathcal{H})},$$

for all  $\delta \in ]0, \infty]$ ,  $f \in \widetilde{W}^{l,p}(\mathcal{H})$  and  $\alpha \in \mathbb{N}_0^n$  with  $|\alpha| = l$ .

*Proof.* The proof immediately follows by Corollary 2.1 and by observing that for all  $f \in \widetilde{W}^{l,p}(\mathcal{H})$  we have that  $\text{supp} f_0 \subset \{x \in \Omega_{\mathcal{H}} : |\rho_n(x)| < D\}$ .  $\square$

### 3. BURENKOV'S EXTENSION OPERATOR ON GENERAL LIPSCHITZ OPEN SETS

We recall the definition of open set with Lipschitz boundary. Here and in the sequel, given a set  $C$  in  $\mathbb{R}^n$  and  $d > 0$  we denote by  $C_d$  the set  $\{x \in C : \text{dist}(x, \partial C) > d\}$ .

**Definition 3.1.** *Let  $d > 0$ ,  $M \geq 0$ ,  $s \in \mathbb{N} \cup \{\infty\}$ . Let  $\{V_j\}_{j=1}^s$  be a family of cuboids, i.e. for every  $j = \overline{1, s}$  there exists an isometry  $\lambda_j$  in  $\mathbb{R}^n$  such that*

$$\lambda_j(V_j) = \Pi_{i=1}^n [a_{i,j}, b_{i,j}[$$

where  $0 < a_{i,j} < a_{i,j} + d < b_{i,j}$ . Assume that  $D := \sup_{j=\overline{1, s}} \text{diam} V_j < \infty$ ,  $(V_j)_d \neq \emptyset$  for all  $j = \overline{1, s}$ , and that the multiplicity of the covering  $\{V_j\}_{j=1}^s$  is finite. We then say that  $\mathcal{A} = (s, d, \{V_j\}_{j=1}^s, \{\lambda_j\}_{j=1}^s)$  is an atlas.

Let  $M \geq 0$ . We say that an open set  $\Omega$  in  $\mathbb{R}^n$  is of class  $C_M^{0,1}(\mathcal{A})$  if the following conditions are satisfied:

(i) For every  $j = \overline{1, s}$ , we have  $\Omega \cap (V_j)_d \neq \emptyset$ .

(ii)  $\Omega \subset \cup_{j=1}^s (V_j)_d$ .

(iii) For every  $j = \overline{1, s}$ , the set  $\mathcal{H}_j := \lambda_j(\Omega \cap V_j)$  satisfies the following condition: either  $\mathcal{H}_j = \Pi_{i=1}^n [a_{i,j}, b_{i,j}[$  (in which case  $V_j \subset \Omega$ ), or  $\mathcal{H}_j$  is a bounded elementary domain with Lipschitz boundary and parameters  $d, D, M$  of the form

$$\mathcal{H}_j = \{x \in \mathbb{R}^n : \bar{x} \in W_j, a_{n,j} < x_n < \varphi_j(\bar{x})\}$$

where  $\varphi_j$  is a real-valued Lipschitz function defined on  $W_j = \Pi_{i=1}^{n-1} [a_{i,j}, b_{i,j}[$  such that

$$a_{n,j} + d < \varphi_j \quad \text{and} \quad \text{Lip} \varphi_j \leq M$$

(in which case  $V_j \cap \partial \Omega \neq \emptyset$ ).

Let  $\mathcal{A} = (s, d, \{V_j\}_{j=1}^s, \{\lambda_j\}_{j=1}^s)$  be an atlas and let  $\mathcal{H}_j = \lambda_j(\Omega \cap V_j)$  for all  $j = \overline{1, s}$ , as above. For every  $j = \overline{1, s}$ , we consider an extension operator  $T_{\mathcal{H}_j}$  from  $\widetilde{W}^{l,p}(\mathcal{H}_j)$  to  $W^{l,p}(\mathbb{R}^n)$  which is the operator defined by (23) if  $V_j \cap \partial \Omega \neq \emptyset$  and is just the extension-by-zero operator if  $V_j \subset \Omega$ .

Next, for every  $j = \overline{1, s}$ , we consider the push-forward operator  $\Lambda_j$  from  $W^{l,p}(\Omega \cap V_j)$  to  $W^{l,p}(\mathcal{H}_j)$  defined by  $\Lambda_j f = f \circ \lambda_j^{(-1)}$  for all  $f \in W^{l,p}(\Omega \cap V_j)$  and we set  $\widetilde{W}^{l,p}(\Omega \cap V_j) = \Lambda_j^{(-1)}(\widetilde{W}^{l,p}(\mathcal{H}_j))$ . Note that



$\widetilde{W}^{l,p}(\Omega \cap V_j)$  is the space of functions in  $W^{l,p}(\Omega \cap V_j)$  such that their support has positive distance from the boundary of  $V_j$ . Moreover, we consider the corresponding pull-back operator defined now from  $W^{l,p}(\mathbb{R}^n)$  to itself, which we call directly  $\Lambda_j^{(-1)}$  and which is defined by  $\Lambda_j^{(-1)}u = u \circ \lambda_j$  for all  $u \in W^{l,p}(\mathbb{R}^n)$ . Finally, we set

$$T_j := \Lambda_j^{(-1)} \circ T_{\mathcal{H}_j} \circ \Lambda_j,$$

and we note that  $T_j$  is a well-defined linear continuous extension operator from  $\widetilde{W}^{l,p}(\Omega \cap V_j)$  to  $W^{l,p}(\mathbb{R}^n)$ .

Following [3, p.265], given an open set  $\Omega$  of class  $C_M^{0,1}(\mathcal{A})$ , we consider a family of functions  $\{\psi\}_{j=1}^s$  such that  $\psi_j \in C_c^\infty(\mathbb{R}^n)$ ,  $\text{supp } \psi_j \subset (V_j)_d$ ,  $0 \leq \psi_j \leq 1$ ,  $\sum_{j=1}^s \psi_j^2(x) = 1$  for all  $x \in \Omega$  and such that  $\|D^\alpha \psi_j\|_{L^\infty(\mathbb{R}^n)} \leq M$  for all  $j = \overline{1, s}$  and  $\alpha \in \mathbb{N}_0^n$  with  $|\alpha| \leq l$ , where  $M$  depends only on  $n, l, d$ .

We are able now to define the Burenkov's Extension Operator  $T$  from  $W^{l,p}(\Omega)$  to  $W^{l,p}(\mathbb{R}^n)$  as follows:

$$(25) \quad Tf = \sum_{j=1}^s \psi_j T_j(f \psi_j),$$

for all  $f \in W^{l,p}(\Omega)$ . Note that  $\text{supp } \Lambda_j(f \psi_j) \subset \Pi_{i=1}^s]a_j, b_j[$ , hence  $T_j(f \psi_j)$  is well-defined.

Before giving the proof of the main result of this section, we need to prove the following lemma.

**Lemma 3.1.** *Let  $l \in \mathbb{N}_0$ ,  $p \in [1, \infty[$  and  $\phi : ]0, +\infty[ \rightarrow ]0, +\infty[$ . Let  $\mathcal{A}$  be an atlas in  $\mathbb{R}^n$ ,  $M \geq 0$  and  $\Omega$  be an open set of class  $C_M^{0,1}(\mathcal{A})$ . Then there exists  $c > 0$  depending only on  $n, \mathcal{A}, M, p$  such that*

$$\|T_j f\|_{M_p^{\phi, \delta}(\mathbb{R}^n)} \leq c \|f\|_{M_p^{\phi, \delta}(\Omega \cap V_j)},$$

and

$$(26) \quad \|D^\alpha T_j f\|_{M_p^{\phi, \delta}(\mathbb{R}^n)} \leq c \sum_{|\beta|=l} \|D^\beta f\|_{M_p^{\phi, \delta}(\Omega \cap V_j)},$$

for all  $\delta \in ]0, \infty]$ ,  $j = \overline{1, s}$ ,  $f \in \widetilde{W}^{l,p}(\Omega \cap V_j)$  and  $\alpha \in \mathbb{N}_0^n$  with  $|\alpha| = l$ .

*Proof.* The proof is an easy consequence of Lemma 2.1. For the convenience of the reader we write a few details for the proof of (26). We assume directly that  $V_j \cap \partial\Omega \neq \emptyset$  since the other case  $V_j \cap \partial\Omega = \emptyset$  is trivial. Let  $\alpha, \delta$  and  $f$  be as in the statement and let  $B_r$  be a ball in  $\mathbb{R}^n$  with radius  $r < \delta$ . By

changing variables in integrals and applying the chain rule, we immediately deduce from Corollary 2.1 that

$$\begin{aligned}
\int_{B_r} |D^\alpha T_j f|^p dx &= \int_{B_r} |D^\alpha (T_{\mathcal{H}_j} \circ \Lambda_j(f)) (\lambda_j(x))|^p dx \\
&= \int_{\lambda_j(B_r)} |D^\alpha (T_{\mathcal{H}_j} \circ \Lambda_j(f)) (y)|^p dy \leq c\phi(r) \sum_{|\beta|=l} \|D^\beta (\Lambda_j(f))\|_{M_p^{\phi,\delta}(\mathcal{H}_j)}^p \\
&\leq c\phi(r) \sum_{|\beta|=l} \|D^\beta f\|_{M_p^{\phi,\delta}(\mathcal{H}_j)}^p,
\end{aligned}$$

which allows to conclude.  $\square$

**Remark 3.1.** We note that in Lemma 3.1, Theorem 2.2, and Corollary 2.1, inequalities (26), (24) and (20) allow to estimate the derivatives of the extended function  $Tf$  of order  $|\alpha| = l$  by means of all derivatives of  $f$  of order  $|\beta| = l$ , and this is valid for all functions  $f$  in the Sobolev space  $W^{l,p}$ . Clearly, Burenkov's extension operator defined for functions in  $W^{l,p}$  works also for functions in  $W^{m,p}$  for any  $m \leq l$ . This implies that all above mentioned inequalities hold also for any  $|\alpha| \leq l$ , provided one replaces in the right-hand sides  $|\beta| = l$  by  $|\beta| = |\alpha|$ .

Finally, we can prove the following

**Theorem 3.1.** Let  $\mathcal{A}$  be an atlas in  $\mathbb{R}^n$ ,  $M > 0$  and  $\Omega$  be an open set of class  $C_M^{0,1}(\mathcal{A})$ . Let  $l \in \mathbb{N}$ ,  $p \in [1, \infty[$  and  $\phi : ]0, +\infty[ \rightarrow ]0, +\infty[$ .

Then for every  $\delta \in ]0, \infty[$  there exists  $c > 0$  depending only on  $n, \mathcal{A}, M, l, p, \delta$  such that inequality (1) holds for all  $f \in W^{l,p}(\Omega)$  and  $|\alpha| \leq l$ .

Moreover, if  $\Omega$  is bounded,  $c$  can be chosen to be independent of  $\delta$ , hence

$$(27) \quad \|D^\alpha T f\|_{M_p^\phi(\mathbb{R}^n)} \leq c \sum_{0 \leq |\beta| \leq |\alpha|} \|D^\beta f\|_{M_p^\phi(\Omega)},$$

for all  $f \in W^{l,p}(\Omega)$  and  $|\alpha| \leq l$ .

*Proof.* Let  $\delta \in ]0, \infty[$  be fixed and  $B_r$  be a ball of radius  $r$  with  $0 < r \leq \delta$ . We set  $\tilde{S} = \{j \in \overline{1, s} : B_r \cap V_j \neq \emptyset\}$ ,  $\tilde{s} = \#\tilde{S}$  and we note that  $\tilde{s}$  is a finite number depending only on  $\mathcal{A}$  and  $\delta$ .

By Lemma 3.1 and Remark 3.1 we get

$$\begin{aligned}
 (28) \quad \int_{B_r} |D^\alpha T f|^p dx &= \int_{B_r} |D^\alpha (\sum_{j=1}^s \psi_j T_j(f \psi_j))|^p dx \\
 &\leq c \sum_{j \in \tilde{S}} \int_{B_r \cap V_j} \sum_{\gamma \leq \alpha} |D^{\alpha-\gamma} \psi_j D^\gamma T_j(f \psi_j)|^p dx \\
 &\leq c \sum_{j \in \tilde{S}} \int_{B_r} \sum_{\gamma \leq \alpha} |D^\gamma T_j(f \psi_j)|^p dx \\
 &\leq c \phi(r) \sum_{j \in \tilde{S}} \sum_{0 \leq |\beta| \leq |\alpha|} \|D^\beta f\|_{M_p^{\phi, \delta}(\Omega \cap V_j)}^p \\
 &\leq c \tilde{s} \phi(r) \sum_{0 \leq |\beta| \leq |\alpha|} \|D^\beta f\|_{M_p^{\phi, \delta}(\Omega)}^p,
 \end{aligned}$$

which implies the validity of (1).

If  $\Omega$  is bounded then  $\tilde{s} \leq s < \infty$ , hence  $\tilde{s}$  in the previous inequality can be replaced by  $s$  which is independent of  $\delta$ , and (27) follows.  $\square$

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